arXiv:math/0309296v4 [math.CT] 16 Mar 2004

Grothendieck categories and support conditions

We give examples of pairs $(\mathcal{G}_1, \mathcal{G}_2)$ where $\mathcal{G}_1$ is a Grothendieck category and $\mathcal{G}_2$ a full Grothendieck subcategory of $\mathcal{G}_1$, the inclusion $\mathcal{G}_2 \hookrightarrow \mathcal{G}_1$ being denoted ι, for which $\mathsf{R}^+\iota : \mathsf{D}^+\mathcal{G}_2 \to \mathsf{D}^+\mathcal{G}_1$ (or even $\mathsf{R}\iota : \mathsf{D}\mathcal{G}_2 \to \mathsf{D}\mathcal{G}_1$) is a a full embedding[1]. This yields generalizations of some results of Bernstein and Lunts, and of Cline, Parshall and Scott. To wit, Theorem 4 (resp. Theorem 6, resp. Theorem 7 and Corollary 8) below strengthen Theorem 17.1 in Bernstein and Lunts [4] (resp. Example 3.3.c and Theorem 3.9.a of Cline, Parshall and Scott [10], resp. Theorem 3.1 and Proposition 3.6 of Cline, Parshall and Scott [9]).

We work in the axiomatic system defined by Bourbaki in [6]. We postulate in addition the existence of an uncountable universe $\mathcal{U}$ in the sense of Bourbaki [7]. All categories are $\mathcal{U}$-categories.

By Alonso Tarrío, Jeremías López and Souto Salorio [1], Theorem 5.4, or by Serpé [19] Theorem 3.13, (or more simply by Spaltenstein [20], proof of Theorem 4.5), the functor[2] $\mathsf{RHom}_{\mathcal{G}_i}$ is defined on the whole of $\mathsf{D}\mathcal{G}_i^{op} \times \mathsf{D}\mathcal{G}_i$. — Consider the following conditions.

(R) : For all $V, W \in \mathsf{D}\mathcal{G}_2$ the complexes $\mathsf{RHom}_{\mathcal{G}_1}(V, W)$ and $\mathsf{RHom}_{\mathcal{G}_2}(V, W)$ are canonically isomorphic in $\mathsf{D}\mathbb{Z}$.

(R+) : For all $V, W \in \mathsf{D}^+\mathcal{G}_2$ the complexes $\mathsf{RHom}_{\mathcal{G}_1}(V, W)$ and $\mathsf{RHom}_{\mathcal{G}_2}(V, W)$ are canonically isomorphic in $\mathsf{D}\mathbb{Z}$.

Let A be a commutative ring, let Y be a set of prime ideals of A, let $\mathcal{G}_1$ (resp. $\mathcal{G}_2$) be the category of A-modules (resp. of A-modules supported on Y). Do (R) or (R+) hold? (See Theorem 3 below for a partial answer.)

By the proof of Weibel [22], Theorem A3, (R) implies (R+) [3]. Moreover, if (R) (resp. (R+)) holds, then $\mathsf{R}\iota$ (resp. $\mathsf{R}^+\iota$) is a full embedding. Indeed we have $\mathsf{Hom}_{\mathsf{D}\mathcal{G}_i} = \mathsf{H}^0\,\mathsf{RHom}_{\mathcal{G}_i}$ (resp. $\mathsf{Hom}_{\mathsf{D}^+\mathcal{G}_i} = \mathsf{H}^0\,\mathsf{RHom}_{\mathcal{G}_i}$) by Lipman [18], I.2.4.2.

Let $\mathsf{Mod}\,A$ denote the category of left A-modules (whenever this makes sense), and let $\mathsf{D}A$ (resp. D^+A, resp. $\mathsf{K}A$, resp. K^+A) be an abbreviation for

[1] The categories $\mathcal{G}_1$ and $\mathcal{G}_2$ will come under various names, but the inclusion will always be denoted by ι.

[2] An example of category for which RHom can be explicitly described is given in Appendix 1.

[3] I know no cases where (R+) holds but (R) doesn't.

$\mathsf{D}\,\mathsf{Mod}\,A$ (resp. $\mathsf{D}^+\,\mathsf{Mod}\,A$, resp. $\mathsf{K}\,\mathsf{Mod}\,A$, resp. $\mathsf{K}^+\,\mathsf{Mod}\,A$), where K means "homotopy category". (Even if $\mathcal{G}_1$ or $\mathcal{G}_2$ is **not** Grothendieck, it may still happen that (R+) or (R) makes sense and holds. In such a situation the phrase "(R+) (resp. (R)) holds" shall mean "(R+) (resp. (R)) makes sense and holds".)

Let $\mathcal{A}$ be a sheaf of rings over a topological space X, let Y be a locally closed subspace of X, let $\mathcal{B}$ be the restriction of $\mathcal{A}$ to Y, and identify, thanks to Section 3.5 of Grothendieck [12], $\mathsf{Mod}\,\mathcal{B}$ to the full subcategory of $\mathcal{A}$-modules supported on Y.

Theorem 1 *The pair* $(\mathsf{Mod}\,\mathcal{A}, \mathsf{Mod}\,\mathcal{B})$ *satisfies (R).*

Proof. Let $r : \mathsf{Mod}\,\mathcal{A} \to \mathsf{Mod}\,\mathcal{B}$ be the restriction functor.

Case 1. Y is closed. — We have for $V \in \mathsf{K}\mathcal{B}$

$$\Big[\mathsf{Hom}^\bullet_{\mathcal{B}}(V,?) = \mathsf{Hom}^\bullet_{\mathcal{A}}(V,?) \circ \mathsf{K}\iota\Big] : \mathsf{K}\mathcal{B} \to \mathsf{K}\mathbb{Z}.$$

Since ι is right adjoint to the exact functor r, it preserves K-injectivity in the sense of Spaltenstein [20]. By Lipman [18], Corollaries I.2.2.7 and I.2.3.2.3, we get

$$\Big[\mathsf{RHom}_{\mathcal{B}}(V,?) \xrightarrow{\sim} \mathsf{RHom}_{\mathcal{A}}(V,?) \circ \mathsf{R}\iota\Big] : \mathsf{D}\mathcal{B} \to \mathsf{D}\mathbb{Z}.$$

Case 2. Y is open. — We have for $V \in \mathsf{K}\mathcal{B}$

$$\Big[\mathsf{Hom}^\bullet_{\mathcal{A}}(V,?) = \mathsf{Hom}^\bullet_{\mathcal{B}}(V,?) \circ \mathsf{K}r\Big] : \mathsf{K}\mathcal{A} \to \mathsf{K}\mathbb{Z}.$$

As r is right adjoint to the exact functor ι, it preserves K-injectivity, and Lipman [18], Corollaries I.2.2.7 and I.2.3.2.3, yields $\mathsf{R}r \circ \mathsf{R}\iota = \mathsf{Id}_{\mathsf{D}\mathcal{B}}$,

$$\Big[\mathsf{RHom}_{\mathcal{A}}(V,?) = \mathsf{RHom}_{\mathcal{B}}(V,?) \circ \mathsf{R}r\Big] : \mathsf{D}\mathcal{A} \to \mathsf{D}\mathbb{Z},$$

and thus

$$\Big[\mathsf{RHom}_{\mathcal{A}}(V,?) \circ \mathsf{R}\iota = \mathsf{RHom}_{\mathcal{B}}(V,?)\Big] : \mathsf{D}\mathcal{B} \to \mathsf{D}\mathbb{Z}. \quad \square$$

Proposition 2 *Let X and $\mathcal{A}$ be as above, let Y be a union of closed subspaces of X, and let $\mathsf{Mod}(\mathcal{A}, Y)$ be the category of $\mathcal{A}$-modules supported on Y. Then the pair $(\mathsf{Mod}\,\mathcal{A}, \mathsf{Mod}(\mathcal{A}, Y))$ satisfies (R+).*

Proof. See Grothendieck [12], Proposition 3.1.2, and Hartshorne [14], Proposition I.5.4. $\square$

Let $(X, \mathcal{O}_X)$ be a noetherian scheme, $\mathcal{A}$ a sheaf of rings over X and $\mathcal{O}_X \to \mathcal{A}$ a morphism, assume $\mathcal{A}$ is $\mathcal{O}_X$-coherent, let Y be a subspace of X, and denote by $\mathsf{QC}\,\mathcal{A}$ (resp. $\mathsf{QC}(\mathcal{A}, Y)$) the category of $\mathcal{O}_X$-quasi-coherent $\mathcal{A}$-modules (resp. $\mathcal{O}_X$-quasi-coherent $\mathcal{A}$-modules supported on Y).

Theorem 3 *The pair $(\mathsf{QC}\,\mathcal{A}, \mathsf{QC}(\mathcal{A}, Y))$ satisfies (R+). If in addition $\mathsf{Ext}^n_{\mathsf{QC}\,\mathcal{A}} = 0$ for $n \gg 0$, then (R) holds*[4].

Let A be a left noetherian ring, let B be a ring, let $A \to B$ be a morphism, let $\mathcal{G}$ be a Grothendieck subcategory of $\mathsf{Mod}\,B$, let $(U_j)_{j\in J}$ be a family of generators of $\mathcal{G}$ which are finitely generated over A, and let I be an Artin-Rees left ideal of A. For each V in $\mathsf{Mod}\,A$ set

$$V_I := \{v \in V \mid I^{n(v)}v = 0 \text{ for some } n(v) \in \mathbb{N}\}. \tag{1}$$

Assume that IV and V_I belong to $\mathcal{G}$ whenever V does. Let $\mathcal{G}_I$ be the full subcategory of $\mathcal{G}$ whose objects satisfy $V = V_I$.

Example: $\mathcal{G}$ is the category of $(\mathfrak{g}, K)$-modules defined in Section 1.1.2 of Bernstein and Lunts [4], A is $U\mathfrak{g}$, B is $U\mathfrak{g} \rtimes \mathbb{C}K$, I is a left ideal of A generated by K-invariant central elements.

Theorem 4 *The pair $(\mathcal{G}, \mathcal{G}_I)$ satisfies (R+). If in addition $\mathsf{Ext}^n_{\mathcal{G}} = 0$ for $n \gg 0$, then (R) holds. In particular if $(\mathcal{G}, \mathcal{G}_I)$ is as in the above Example and if K is reductive, then (R) is fulfilled.*

Lemma 5 *If E is an injective object of $\mathcal{G}$, then so is E_I.*

Lemma 5 implies Theorem 4. By Theorem 1.10.1 of Grothendieck in [12], $\mathcal{G}$ and $\mathcal{G}_I$ have enough injectives. We have for $V \in \mathsf{K}^+\mathcal{G}_I$

$$\Big[\mathsf{Hom}^\bullet_{\mathcal{G}_I}(V, ?) = \mathsf{Hom}^\bullet_{\mathcal{G}}(V, ?) \circ \mathsf{K}^+\iota\Big] : \mathsf{K}^+\mathcal{G}_I \to \mathsf{K}\mathbb{Z}$$

and thus, by Lemma 5 and Hartshorne [14], Proposition I.5.4.b,

$$\Big[\mathsf{RHom}_{\mathcal{G}_I}(V, ?) \xrightarrow{\sim} \mathsf{RHom}_{\mathcal{G}}(V, ?) \circ \mathsf{R}^+\iota\Big] : \mathsf{D}^+\mathcal{G}_I \to \mathsf{D}\mathbb{Z}.$$

This proves the first sentence of the theorem. For the second one the argument is the same except for the fact we use Hartshorne [14], proof of Corollary I.5.3.γ.b. (By the first sentence, $\mathsf{Ext}^n_{\mathcal{G}} = 0$ for $n \gg 0$ implies $\mathsf{Ext}^n_{\mathcal{G}_I} = 0$ for $n \gg 0$.) $\square$

[4]We regard $\mathsf{Ext}^n_{\mathcal{G}}$ as a functor defined on $\mathcal{G}^{op} \times \mathcal{G}$ (and of course **not** on $\mathsf{D}\mathcal{G}^{op} \times \mathsf{D}\mathcal{G}$).

Proof of Lemma 5. Let $W \subset V$ be objects of $\mathcal{G}$ and $f : W \to E_I$ a morphism. We must extend f to $g : V \to E_I$. We can assume, by the proof of Grothendieck [12] Section 1.10 Lemma 1, (or by Stenström [21], Proposition V.2.9), that V is finitely generated over A. Since W is also finitely generated over A, there is an n such that $I^n f(W) = 0$, and thus $f(I^n W) = 0$. Choose a k such that $W \cap I^k V \subset I^n W \subset \mathsf{Ker}\, f$ and set

$$\overline{V} := \frac{V}{I^k V} \quad , \qquad \overline{W} := \frac{W}{W \cap I^k V} \quad .$$

Then f induces a morphism $\overline{W} \to E_I$, which, by injectivity of E, extends to a morphism $\overline{V} \to E$, that in turn induces a morphism $\overline{V} \to E_I$, enabling us to define g as the obvious composition $V \to \overline{V} \to E_I$. $\square$

Let $\mathfrak{g}$ be a complex semisimple Lie algebra, let $\mathfrak{h} \subset \mathfrak{b}$ be respectively Cartan and Borel subalgebras of $\mathfrak{g}$, put $\mathfrak{n} := [\mathfrak{b}, \mathfrak{b}]$, say that the roots of $\mathfrak{h}$ in $\mathfrak{n}$ are positive, let $\mathcal{W}$ be the Weyl group equipped with the Bruhat ordering, let $\mathcal{O}_0$ be the category of those BGG-modules which have the generalized infinitesimal character of the trivial module. The simple objects of $\mathcal{O}_0$ are parametrized by $\mathcal{W}$. Say that $Y \subset \mathcal{W}$ is an **initial segment** if $x \leq y$ and $y \in Y$ imply $x \in Y$, and that $w \in \mathcal{W}$ lies in the **support** of $V \in \mathcal{O}_0$ if the simple object attached to w is a subquotient of V. For such an initial segment Y let $\mathcal{O}_Y$ be the subcategory of $\mathcal{O}_0$ consisting of objects supported on $Y \subset \mathcal{W}$.

Theorem 6 *The pair $(\mathcal{O}_0, \mathcal{O}_Y)$ satisfies (R).*

Proof. In view of BGG [3] this will follow from Theorem 9. $\square$

Let A be a ring, I an ideal, and $B := A/I$ the quotient ring.

Theorem 7 *Assume that $\mathsf{Ext}^n_A(B, B)$ vanishes for $n > 0$, and that there is a p such that $\mathsf{Ext}^n_A(B, W) = 0$ for all $n > p$ and all B-modules W. Then the pair $(\mathsf{Mod}\, A, \mathsf{Mod}\, B)$ satisfies (R).*

Proof.

Step 1 : $\mathsf{Ext}^n_A(B, W) = 0$ for all B-modules W and all $n > 0$. — By Theorem V.9.4 in Cartan-Eilenberg [8] we have $\mathsf{Ext}^n_A(B, F) = 0$ for all free B-modules F and all $n > 0$. Suppose by contradiction there is an $n > 0$ such that $\mathsf{Ext}^n_A(B, ?)$ does not vanish on all B-modules; let n be maximum for this property; choose a B-module V such that $\mathsf{Ext}^n_A(B, V) \neq 0$; consider an exact sequence $W \rightarrowtail F \twoheadrightarrow V$ with F free; and observe the contradiction $0 \neq \mathsf{Ext}^n_A(B, V) \xrightarrow{\sim} \mathsf{Ext}^{n+1}_A(B, W) = 0$.

Step 2 : Putting $r := \mathsf{Hom}_A(B, ?)$ we have $\mathsf{R}r \circ \mathsf{R}\iota = \mathsf{Id}_{\mathsf{D}B}$. — The functor r, being a right adjoint, commutes with products, and, having an exact left adjoint, preserves injectives. Let V be in $\mathsf{D}B$ and I a Cartan-Eilenberg injective resolution (CEIR) of V in $\mathsf{Mod}\,A$. By the previous step rI is a CEIR of $rV = V$ in $\mathsf{Mod}\,B$. Weibel [22], Theorem A3, implies

(a) the complex $\mathsf{Tot}^\Pi I \in \mathsf{D}A$, characterized by

$$(\mathsf{Tot}^\Pi I)^n = \prod_{p+q=n} I^{pq},$$

is a K-injective resolution (see Spaltenstein [20]) of V in $\mathsf{Mod}\,A$,

(b) $\mathsf{Tot}^\Pi rI = r\mathsf{Tot}^\Pi I$ is a K-injective resolution of $V = rV$ in $\mathsf{Mod}\,B$.

Statement (a) yields: (c) $r\mathsf{Tot}^\Pi I = \mathsf{R}rV$. Then (b) and (c) imply that the natural morphism $V \to \mathsf{R}rV$ is a quasi-isomorphism.

Step 3 : (R) holds. — See proof of Theorem 1, Case 2. $\square$

Corollary 8 *If there is a projective resolution $P = (P_n \rightarrowtail \cdots \to P_1 \to P_0)$ of B by A-modules satisfying $\mathsf{Hom}_A(P_j, V) = 0$ for all B-modules V and all $j > 0$, then pair $(\mathsf{Mod}\,A, \mathsf{Mod}\,B)$ satisfies (R).*

Let A be a ring, X a finite set and $e_\bullet = (e_x)_{x \in X}$ a family of idempotents of A satisfying $\sum_{x\in X} e_x = 1$ and $e_x e_y = \delta_{xy} e_x$ (Kronecker delta) for all $x, y \in X$.

The **support** of an A-module V is the set $\{x \in X \mid e_x V \neq 0\}$. Let $\leq$ be a partial ordering on X, and for any initial segment Y put

$$A_Y := A \Big/ \sum_{x \notin Y} A e_x A \ ,$$

so that $\mathsf{Mod}\,A_Y$ is the full subcategory of $\mathsf{Mod}\,A$ whose objects are supported on Y. (Here and in the sequel, for any ring B, we denote by BbB the ideal generated by $b \in B$.) The image of e_y in A_Y will be still denoted by e_y.

Assume that, for any pair (Y, y) where Y is an initial segment and y a maximal element of Y, the module $M_y := A_Y e_y$ does **not** depend on Y, but only on y. This is equivalent to the requirement that $A_Y e_y$ be supported on $\{x \in X \mid x \leq y\}$.

If $(V_\gamma)_{\gamma\in\Gamma}$ a family of A-modules, let $\langle V_\gamma\rangle_{\gamma\in\Gamma}$ denote the class of those A-modules which admit a finite filtration whose associated graded object is isomorphic to a product of members of the family.

Assume that, for any $x \in X$, the module Ae_x belongs to $\langle M_y\rangle_{y\in X}$.

Theorem 9 *The pair* $(\mathsf{Mod}\, A, \mathsf{Mod}\, A_Y)$ *satisfies (R).*

This statement applies to the categories satisfying Conditions (1) to (6) in Section 3.2 of Beilinson, Ginzburg and Soergel [2], like the categories of BGG modules $\mathcal{O}_\lambda$ and $\mathcal{O}^{\mathfrak{q}}$ defined in Section 1.1 of [2], or more generally the category $\mathcal{P}(X, \mathcal{W})$ of perverse sheaves considered in Section 3.3 of [2]. — Because of the projectivity of $M_x = Ae_x$ we have

Lemma 10 *For any $x, y \in X$ with x maximal there is a nonnegative integer n and an exact sequence $(Ae_x)^n \rightarrowtail Ae_y \twoheadrightarrow V$ such that $V \in \langle M_z\rangle_{z<x}$. In particular $e_xV = 0$.* $\square$

Proof of Theorem 9. Assume $Y = X\backslash\{x\}$ where x is maximal. Put $e := e_x$, $I := AeA$ and $B := A_Y = A/I$. By the previous Lemma there is a nonnegative integer n and an exact sequence $(Ae)^n \rightarrowtail A \twoheadrightarrow V$ with $IV = 0$. Letting $J \subset A$ be the image of $(Ae)^n \rightarrowtail A$, we have $J = IJ \subset I \subset J$, and thus $I = J$. In particular I is A-projective and we have $\mathsf{Hom}_A(I, B) \simeq (eB)^n = 0$. Corollary 8 applies, proving Theorem 9 for the particular initial segment Y. Lemma 10 shows that $(B, Y, (e_y)_{y\in Y})$ satisfies the assumptions of Theorem 9, and an obvious induction completes the proof. $\square$

For any complex Lie algebra $\mathfrak{g}$ let $I_{\mathfrak{g}}$ be the annihilator of the trivial module in the center of the enveloping algebra. Using the notation and definitions of Knapp and Vogan [17], let $(\mathfrak{g}, K)$ be a reductive pair, let $(\mathfrak{g}', K')$ be a reductive subpair attached to θ-stable subalgebra, let $\mathcal{R}^S : \mathcal{C}(\mathfrak{g}', K') \to \mathcal{C}(\mathfrak{g}, K)$ be the cohomological induction functor defined in [17], (5.3.b), and let $\mathcal{G}$ (resp. $\mathcal{G}'$) be the category of $(\mathfrak{g}, K)$-modules on which $I_{\mathfrak{g}}$ (resp. $I_{\mathfrak{g}'}$) acts locally nilpotently. By [17], Theorem 11.225, the functor $\mathcal{R}^S$ maps $\mathcal{G}'$ to $\mathcal{G}$. Let $F : \mathcal{G}' \to \mathcal{G}$ be the induced functor. By [17], Theorem 3.35.b, F is exact. It would be interesting to know if F satisfies Condition (R).

Thank you to Anton Deitmar, Bernhard Keller and Wolfgang Soergel for their interest, and to Martin Olbrich for having pointed out some mistakes in a previous version.

Proof of Theorem 3

Put $\mathcal{O} := \mathcal{O}_X$ and consider the following statements:

(a) Every object of $\mathsf{QC}(\mathcal{O}, Y)$ is contained into an object of $\mathsf{QC}(\mathcal{O}, Y)$ which is injective in $\mathsf{QC}\,\mathcal{O}$.

(b) Every object of $\mathsf{QC}(\mathcal{A}, Y)$ is contained into an object of $\mathsf{QC}(\mathcal{A}, Y)$ which is injective in $\mathsf{QC}\,\mathcal{A}$.

We claim (a) $\Longrightarrow$ (b) $\Longrightarrow$ Theorem 3.

(a) $\Longrightarrow$ (b) : The functor $\mathcal{H}om_{\mathcal{O}}(\mathcal{A}, ?)$ preserves the following properties:

- quasi-coherence (by EGA I [13], Corollary 2.2.2.vi),
- the fact of being supported on Y (by Grothendieck [12], Proposition 4.1.1),
- injectivity (by having an exact left adjoint). $\square$

(b) $\Longrightarrow$ Theorem 3 : See proof of Theorem 4. $\square$

Proof of (a). Let M be in $\mathsf{QC}(\mathcal{O}, Y)$ and let us show that M is contained into an object of $\mathsf{QC}(\mathcal{O}, Y)$ which is injective in $\mathsf{QC}\,\mathcal{O}$. We may, and will, assume that Y is precisely the support of M.

Case 1. M is coherent, $(X, \mathcal{O})$ is affine. — Write A for $\Gamma\mathcal{O}$, where Γ is the global section functor. Use the equivalence $\mathsf{QC}\,\mathcal{O} \xrightarrow{\sim} \mathsf{Mod}\,A$ set up by Γ to work in the latter category. Then M "is" a finitely generated A-module, and Y is closed by Proposition II.4.4.17 in Bourbaki [5]. Let $I \subset A$ be the ideal of those f in A which vanish on Y, and $\mathsf{Mod}(A, Y)$ the full subcategory of $\mathsf{Mod}\,A$ whose objects are the A-modules V satisfying $V = V_I$ in the sense of Notation (1). Corollary 2 to Proposition II.4.4.17 in Bourbaki [5] implies that Γ induces a subequivalence $\mathsf{QC}(\mathcal{O}, Y) \xrightarrow{\sim} \mathsf{Mod}(B, Y)$. The claim now follows from Theorem 4.

Case 2. M is coherent. — Argue as in the proof of Corollary III.3.6 in Hartshorne [15], using Proposition 6.7.1 of EGA I [13].

Case 3. General case. — By Gabriel [11] Corollary 1 §II.4 (p. 358), Theorem 2 §II.6 (p. 362), and Theorem 1 §VI.2 (p. 443) we know that every object of $\mathsf{QC}\,\mathcal{O}$ has an injective hull and that any colimit of injective objects of $\mathsf{QC}\,\mathcal{O}$ is injective. The expression $M \prec M'$, shall mean "M' is an injective hull of M and $M \subset M'$". Let M' be such a hull and Z the set of pairs (N, N') with

$$N \subset M, \quad N' \subset M', \quad N \prec N', \quad \mathsf{Supp}(N') = \mathsf{Supp}(N).$$

Then Z, equipped with its natural ordering, is inductive. Let (N, N') is a maximal element of Z and suppose by contradiction $N \neq M$. By Corollary 6.9.9 of EGA I [13] there is a P such that $N \subset P \subset M$, $N \neq P$, and $C := P/N$ is coherent. Let $\pi : P \twoheadrightarrow C$ be the canonical projection and choose P', C' such that $P \prec P', C \prec C'$. By injectivity of N' there is a map $f : P \to N'$ such that $[N \hookrightarrow P \xrightarrow{f} N'] = [N \hookrightarrow N']$ (obvious notation). Consider the commuting diagram

$$\begin{tikzcd}
N' \arrow[r, hook] & N' \times C' & C' \arrow[l, hook'] \\
N \arrow[u, hook] \arrow[r, hook] & P \arrow[u, "f\times\pi"'] \arrow[r, two heads, "\pi"] & C. \arrow[u, hook]
\end{tikzcd}$$

We have $\mathsf{Ker}(f \times \pi) = \mathsf{Ker}(f) \cap \mathsf{Ker}(\pi) = \mathsf{Ker}(f) \cap N = 0$, i.e. $g := f \times \pi$ is monic. By injectivity of $N' \times C'$ there is a map $P' \to N' \times C'$ such that $[P \hookrightarrow P' \to N' \times C'] = [P \overset{g}{\rightarrowtail} N' \times C']$, this map being monic by essentiality of $P \subset P'$; in particular

$$\mathsf{Supp}(P') \subset \mathsf{Supp}(N') \cup \mathsf{Supp}(C').$$

A similar argument shows the existence of a monomorphism $P' \rightarrowtail M'$ such that $[P \hookrightarrow P' \rightarrowtail M'] = [P \hookrightarrow M \hookrightarrow M']$, meaning that we can assume $P' \subset M'$. Since $(P, P') \notin Z$, this implies $\mathsf{Supp}(P) \neq \mathsf{Supp}(P')$, and the equalities

$$\mathsf{Supp}(N') = \mathsf{Supp}(N) \quad \text{(because } (N, N') \in Z),$$

$$\mathsf{Supp}(C') = \mathsf{Supp}(C) \quad \text{(by Case 2)},$$

yield the contradiction

$$\mathsf{Supp}(P') \subset \mathsf{Supp}(N) \cup \mathsf{Supp}(C) = \mathsf{Supp}(P) \subset \mathsf{Supp}(P'). \ \square$$

Appendix 1

Let k be a field and $\mathfrak{g}$ a Lie k-algebra. For $X, Y \in \mathsf{D}k$ put

$$\langle X, Y \rangle := \mathsf{Hom}_k^\bullet(X, Y).$$

Let $C := U\mathfrak{g} \otimes \bigwedge \mathfrak{g}$ be the Koszul complex viewed as a differential graded coalgebra (here and in the sequel tensor products are taken over k).

In view of Weibel [22], Theorem A3, we can define $\mathsf{RHom}_{\mathfrak{g}}$ by setting

$$\mathsf{RHom}_{\mathfrak{g}}(X, Y) := \langle\langle C, X\rangle, \langle C, Y\rangle\rangle^{\mathfrak{g}}.$$

(As usual the superscript $\mathfrak{g}$ means "$\mathfrak{g}$-invariants".) Recall that the Chevalley-Eilenberg complex, used to compute the cohomology of $\mathfrak{g}$ with values in $\langle X, Y\rangle$, is defined by $\mathsf{CE}(X,Y) := \langle C, \langle X, Y\rangle\rangle^{\mathfrak{g}}$, and that there is a canonical isomorphism $F : \mathsf{CE} \xrightarrow{\sim} \mathsf{RHom}_{\mathfrak{g}}$. Let

$$\mathsf{ext}_{X,Y,Z} : \mathsf{CE}(Y,Z) \otimes \mathsf{CE}(X,Y) \to \mathsf{CE}(X,Z)$$

be the exterior product and

$$\mathsf{comp}_{X,Y,Z} : \mathsf{RHom}_{\mathfrak{g}}(Y,Z) \otimes \mathsf{RHom}_{\mathfrak{g}}(X,Y) \to \mathsf{RHom}_{\mathfrak{g}}(X,Z)$$

the composition. Then the expected formula

$$\mathsf{comp}_{X,Y,Z} \circ (F_{Y,Z} \otimes F_{X,Y}) = F_{X,Z} \circ \mathsf{ext}_{X,Y,Z}$$

is easy to check.

Appendix 2

The following fact is used in various places (see for instance the proofs of Theorem I.3.3 in Cartan-Eilenberg [8], Theorem 1.10.1 in Grothendieck [12] and Lemma 4.3 in Spaltenstein [20]). We use the notation and definitions of Jech [16].

Lemma 11 *Let P be a poset, α a cardinal $\geq |P|$, and β the least cardinal $> \alpha$. Then every poset morphism $f : \beta \to P$ is stationary.*

Proof. We can assume P is infinite and f is epic. The morphism $g : P \to \beta$ defined by $gp := \min f^{-1}p$ satisfies $fg = \mathsf{Id}_P$. Put $\sigma := \sup gP$. For all $p \in P$ we have $|gp| \leq gp < \beta$, implying $|gp| \leq \alpha$ for all p, and $\sigma \leq \beta$. Statement (2.4) and Theorem 8 in Jech [16] entail respectively $\sigma = \bigcup_{p\in P} gp$ and $|P|\,\alpha = \alpha$, from which we conclude $|\sigma| \leq \alpha$; this forces $\sigma < \beta$, that is $\sigma \in \beta$. For any $\gamma \in \beta$, $\gamma > \sigma$ we have $f\gamma = fgf\gamma \leq f\sigma \leq f\gamma$. $\square$

References

[1] Alonso Tarrío L., Jeremías López A., Souto Salorio M. J.; Localization in categories of complexes and unbounded resolutions, *Canad. J. Math.* **52** (2000), no. 2, 225–247,

http://web.usc.es/˜lalonso/paginas-en/publi-en.html

[2] Beilinson A., Ginzburg V., Soergel W.; Koszul duality patterns in representation theory, *J. Am. Math. Soc.* **9** (1996) No.2, 473-527,

http://home.mathematik.uni-freiburg.de/soergel/#Preprints

http://www.ams.org/journal-getitem?pii=S0894-0347-96-00192-0

[3] Bernstein I.N., Gelfand I.M., Gelfand S.I.; Category of $\mathfrak{g}$-modules, *Funct. Anal. Appl.* **10** (1976) 87-92.

[4] Bernstein J., Lunts V.; Localization for derived categories of $(\mathfrak{g}, K)$-modules, *J. Amer. Math. Soc.* **8** (1995), no. 4, 819-856.

[5] Bourbaki N.; **Algèbre commutative**, Chapitres 1 à 4, Hermann, Paris, 1961.

[6] Bourbaki N.; **Théorie des ensembles**, Paris, Hermann 1970.

[7] Bourbaki N.; Univers, Appendix to Exposé I, SGA 4, tome 1, Lect. Notes in Math. 269, Springer 1972, 185-217,

http://modular.fas.harvard.edu/sga/

[8] Cartan H., Eilenberg S.; **Homological algebra**, Princeton University Press, 1956.

[9] Cline E., Parshall B., Scott L.; Algebraic stratification in representation categories, *J. Algebra* **117** (1988), no. 2, 504-521.

[10] Cline E., Parshall B., Scott L.; Finite dimensional algebras and highest weight categories, *J. Reine Angew. Math.* **391** (1988) 85-99.

[11] Gabriel P.; Des catégories abéliennes, *Bull. Soc. Math. France* **90** (1962) 323-448,

http://www.numdam.org/item?id=BSMF_1962__90__323_0

[12] Grothendieck A.; Sur quelques points d'algèbre homologique, *Tôhoku Math. J. (2)* **9** (1957) 119-221.

[13] Grothendieck A., Dieudonné J.; **Éléments de géométrie algébrique**, Springer, 1971.

[14] Hartshorne R.; **Residues and duality**, Lect. Notes in Math. 20, Springer, 1966.

[15] Hartshorne R.; **Algebraic geometry**, Springer, 1977.

[16] Jech T.; **Set theory**, Academic Press, 1978.

[17] Knapp A., Vogan D., **Cohomological induction and unitary representations**, Princeton University Press (1995).

[18] Lipman J.; **Notes on Derived Categories and Derived Functors**,

http://www.math.purdue.edu/~lipman/

[19] Serpé C.; Resolution of unbounded complexes in Grothendieck categories. *J. Pure Appl. Algebra* **177**, No.1, (2003), 103-112.

http://www.sciencedirect.com/

http://wwwmath.uni-muenster.de/math/inst/sfb/about/publ/serpe02.html

[20] Spaltenstein N.; Resolutions of unbounded complexes, *Compositio Math.* **65** (1988), no. 2, 121-154.

[21] Stenström B.; **Rings of quotients**, Springer, 1975.

[22] Weibel Ch.; Cyclic homology for schemes, *Proc. AMS* **124** (1996) 1655-1662,

http://www.math.uiuc.edu/K-theory/0043/

http://www.ams.org/journal-getitem?pii=S0002-9939-96-02913-9

March 7, 2004

Pierre-Yves Gaillard, Département de Mathématiques, Université Nancy 1, France
http://www.iecn.u-nancy.fr/~gaillard/